Math 10B with Professor Stankova
Worksheet, Discussion #2; Tuesday, 1/23/2018
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Cartesian Product

Complementary Counting and PIE

Examples

1. Out of 200 students, there are 100 taking Calculus, 70 taking algebra, and 30 taking both. How many students are taking neither?

> **Solution:** There are $100+70-30=140$ students taking Calculus or algebra. Thus, there are $200-140=60$ taking neither.

2. How many 4 letter sequences do not contain the same letter twice in a row?

> **Solution:** There are a total of 26^4 4 letter sequences. Then we count the number of ways for the sequence to have at least two in a row. If these two in the row occur in the first two, this gives 26 for the selection of the repeated letter, then 26^2 for the other two letters. Thus we get 26^3 and the same holds true if the two repeated letters are in the middle or at the end. Then we add back the intersections which are the cases $AAAB, AABB, ABBB$ and all three give 26^3, and taking the intersection of those give 26. So the final answer is
>
> $$26^4 - 3\cdot 26^3 + 3\cdot 26^2 - 26.$$

Problems

3. True **FALSE** Complementary counting is not related to Principle of Inclusion/Exclusion.

> **Solution:** Complementary counting is like using PIE with a Venn diagram with only one circle.

4. True **FALSE** Venn diagrams with two circles always look like interlocking rings.

> **Solution:** A circle inside another is a Venn diagram.

5. How many ways are there to put 7 balls in 3 boxes if each box must have at least one ball?

> **Solution:** There are 3^7 ways to put the 7 balls in 3 boxes. Let A, B, C be the cases where box $1, 2, 3$ are empty respectively. Then the number of ways for each is 2^7 and the intersections happen in a unique way. Thus, the total number of ways is
>
> $$3^7 - 3 \cdot 2^7 + 3.$$

6. How many numbers from 1 to 300 are even but not divisible by 3?

> **Solution:** There are 150 even numbers at 50 that are divisible by 6. Thus, there are $150 - 50 = 100$ that are even but not divisible by 3.

7. Last semester, out of all the students who took both intro chem and 10A, 75% of students passed the intro chem final and 85% passed the 10A final and 70% passed both. There were 50 students who failed both. How many total students took both intro chem and 10A?

> **Solution:** There are $75 + 85 - 70 = 90\%$ of students who passed at least one of the finals. Thus, there are $100 - 90 = 10\%$ that failed. So 50 students is 10% and hence there are 500 students.

8. How many numbers less than or equal to 1000 are divisible by 7 or 11 but not both?

> **Solution:** There are $\lfloor 1000/7 \rfloor = 142$ numbers divisible by 7. There are $\lfloor 1000/11 \rfloor = 90$ numbers divisible by 11. There are $\lfloor 1000/77 \rfloor = 12$ numbers divisible by both. Thus, $142 + 90 - 2 \cdot 12 = 208$ are divisible by 7 or 11 but not both.

9. How many license plates with 3 letters followed by 3 digits have either the 3 letters forming a palindrome or the 3 digits forming a palindrome (or both)?

Solution: There are $26^2 \cdot 10^3$ plates with a palindrome letter set and $26^3 \cdot 10^2$ with a palindrome number and $26^2 \cdot 10^2$ with both. Thus, there are $26^2 \cdot 10^3 + 26^3 \cdot 10^2 - 26^2 \cdot 10^2 = 26^2 \cdot 10^2(26 + 10 - 1) = 35 \cdot 26^2 \cdot 10^2$ different plates.

10. How many four digit numbers do not have any repeating 1s?

Solution: There are a total of $9 \cdot 10^3$ four digit numbers. The bad cases are $11XX, X11X, XX11$. The cases have $100, 90, 90$ possibilities. The intersections are $111X, 1111, X111$ which give $10, 1, 9$ cases. Finally the intersection of all three is 1111 which has 1 case. Thus the total number is

$$9000 - 100 - 90 - 90 + 10 + 1 + 9 - 1.$$

11. (Challenge) How many ways can we choose non-empty subsets $A, B \subset \{1, 2, 3, 4, 5\}$ such that $A \cap B = \emptyset$.

Solution: Each number 1 through 5 can either be in A or B or neither. Thus, there are 3^5 different ways to have these subsets. But, there 2^5 ways for A to be empty, 2^5 for B to be empty. And finally 1 way for both to be empty. Thus, there are a total of $3^5 - 2 \cdot 2^5 + 1$ different ways.

Pigeonhole Principle

12. I have 7 pairs of socks in my drawer, one of each color of the rainbow. How many socks do I have to draw out in order to guarantee that I have grabbed at least one pair? What if there are likewise colored pairs of gloves in there and I cannot tell the difference between gloves and socks and I want a matching set?

Solution: After grabbing 7 socks, worst case scenario, I have grabbed a sock of each color. Thus, after grabbing one more sock, it has to match up with one of the previous socks so after grabbing 8 socks I am guaranteed to have a pair.

For the second part, after grabbing 21 objects, it is possible that I have grabbed 3 items for each color and hence have gotten no sets yet. But the 22nd thing I grab must complete one of these 7 sets so after 22 items, I am guaranteed to have a matching set.